

# Planar digraphs of digirth four are 2-colourable

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## Abstract

Neumann-Lara conjectured in 1985 that every planar digraph with digirth at least three is 2-colourable, meaning that the vertices can be 2-coloured without creating any monochromatic directed cycles. We prove a relaxed version of this conjecture: every planar digraph of digirth at least four is 2-colourable.

**Keywords:** Planar digraph, digraph chromatic number.

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# 1 Introduction

Let  $D$  be an oriented graph (i.e., a digraph without cycles of length at most 2). A function  $f : V(D) \rightarrow \{1, \dots, k\}$  is a  $k$ -colouring of  $D$  if the subdigraph induced by vertices of colour  $i$  is acyclic for all  $i$ . We say that  $D$  is  $k$ -colourable if it admits a  $k$ -colouring.

The following conjecture was proposed by Neumann-Lara [4] (and independently by Škrekovski, see [1]).

**Conjecture 1.1.** *Every oriented planar graph is 2-colourable.*

There seems to be a lack of methods to attack Conjecture 1.1, and only sporadic partial results are known.

The *digirth* of a digraph is the length of its shortest directed cycle. Harutyunyan and one of the authors [2] proved Conjecture 1.1 under additional assumption that the digirth of  $D$  is at least five. Their proof is based on elaborate use of the (nowadays standard) discharging technique. However, it is unlikely that the same method can be pushed further when directed 4-cycles are allowed.

The main result of this note is the following theorem whose proof is based on a novel technique, at least when colourings of graphs are concerned.

**Theorem 1.2.** *Every oriented planar graph with a vertex  $v_0$  such that each directed cycle of length 3 uses  $v_0$  is 2-colourable.*

**Corollary 1.3.** *Every planar digraph with digirth at least 4 is 2-colourable.*

The rest of the paper is devoted to the proof of Theorem 1.2.

## 2 Proof of Theorem 1.2

The main tool that we will use in our proof is the notion of a *Tutte path*. This is a special kind of a path that was first used by Tutte [6, 7] in his proof that every 4-connected planar graph is hamiltonian. A version used in this paper is taken from [5].

Tutte paths are defined using the following notion of connectivity. If  $G$  is a graph and  $H$  is a subgraph of  $G$ , then an  $H$ -component  $B$  of  $G$  is either an edge  $e \in E(G) \setminus E(H)$  with both ends in  $H$  or it is a connected component  $Q$  of  $G - V(H)$  together with all edges from  $Q$  to  $H$  and all ends of these

edges. The vertices of  $V(B) \cap V(H)$  are called the *vertices of attachment* of  $B$ . A bridge with  $k$  vertices of attachment is said to be *k-attached*.

Let  $G$  be a graph with cycle  $C$  and let  $u, v$  be two vertices in  $G$ . A path  $P$  in  $G$  from  $v$  to  $u$  is called a *Tutte path* with respect to  $C$  if

- (i) each  $P$ -component has at most three vertices of attachment and
- (ii) each  $P$ -component containing an edge of  $C$  has at most two vertices of attachment.

**Theorem 2.1** (Thomassen [5]). *Let  $G$  be a 2-connected plane graph with facial cycle  $C$ . Let  $v$  and  $e$  be a vertex and edge, respectively, of  $C$  and let  $u$  be any vertex of  $G$  distinct from  $v$ . Then  $G$  has a Tutte path with respect to  $C$  from  $u$  to  $v$  that contains the edge  $e$ .*

**Lemma 2.2.** *Let  $D$  and  $D'$  be digraphs, whose intersection is a tournament  $T$ . Then any colouring of  $D$  and any colouring of  $D'$  that agree on  $V(T)$  form a colouring of  $D \cup D'$ .*

*Proof.* The only detail to verify is that the combined colouring does not produce a monochromatic cycle. Since the colourings of  $D$  and  $D'$  have no monochromatic cycles, such a directed cycle  $C$  would be composed of  $2r$  ( $r \geq 1$ ) directed paths  $P_1 \cup P_2 \cup \dots \cup P_{2r}$ , where each  $P_i$  is a directed path completely contained in either  $D$  or  $D'$  (joining vertices  $v_i$  and  $v_{i+1}$  in  $T$  ( $i = 1, \dots, r$ , indices taken modulo  $r$ )). Since none of these paths together with the arcs of the tournament on  $T$  forms a directed cycle,  $T$  contains the arcs  $v_i v_{i+1}$ , which all together form a monochromatic directed cycle in  $T$  and hence in both  $D$  and  $D'$ . This contradiction completes the proof.  $\square$

**Lemma 2.3.** *Let  $G$  be a triangulation of the plane with the outer face bounded by a triangle  $T = abc$ . Then for any orientation  $D$  of  $G$  with digirth at least 4 and any precolouring of  $T$  with colours 1 and 2, there exists a 2-colouring of  $D$  that extends the precolouring on  $T$ .*

The first case of the proof of this lemma is similar to that of Wu for induced forests [8].

*Proof.* The proof is by induction, with the base of induction corresponding to the case when the triangulation is 4-connected. We consider the dual graph  $H$  of  $G$ . Note that  $H$  is a cubic graph and that it is cyclically 4-edge-connected, i.e., any 3-edge-cut in  $H$  isolates a single vertex from the rest of the graph. We distinguish two cases, depending on whether the precolouring on  $T$  uses just one or both colours (see Figure 1).

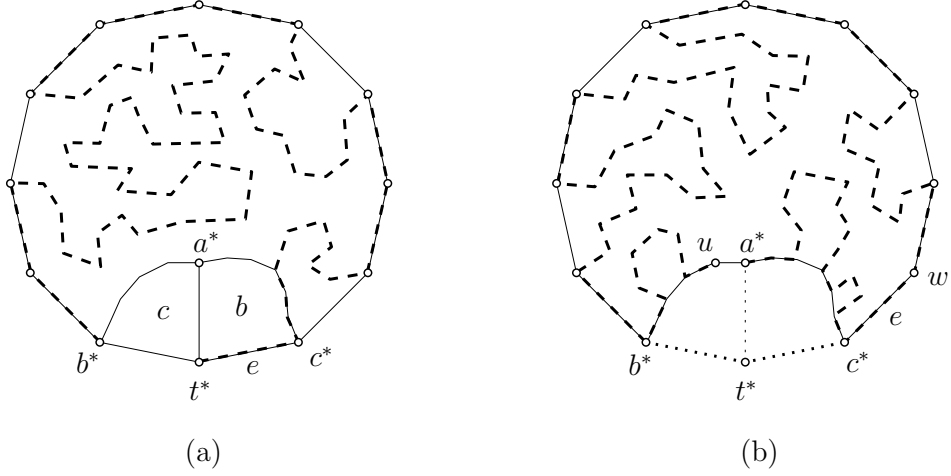


Figure 1: Finding Tutte path  $P^*$  in the 4-connected case. (a) Bicoloured triangle  $abc$ . The dual graph  $H$  is shown, where  $a, b, c$  are faces. (b) Monochromatic triangle  $abc$ .

**4-connected – two colours case.** Suppose first that not all vertices of  $T$  are precoloured the same. We may assume that  $a$  is coloured 1 and that  $b, c$  are coloured 2. Let us denote the vertices of  $H$  corresponding to  $T$  and to the faces surrounding  $T$  by  $t^*, a^*, b^*, c^*$ , where the face  $a^*$  of  $G$  contains vertices  $b, c$  but not  $a$ , etc. Let  $P^*$  be a Tutte path in  $H$  with respect to the facial cycle of  $H$  containing  $b^*, t^*, c^*$  (i.e., the face dual to the vertex  $a$ ) that connects  $t^*$  and  $b^*$  and passes through the edge  $e = t^*c^*$ . This path, whose existence follows from Theorem 2.1, together with the edge  $b^*t^*$  forms a cycle  $C^*$  in  $H$ . The cycle crosses the edges  $ac$  and  $ab$  of  $T$ . We may assume that the exterior of  $C^*$  contains  $b$  and  $c$ , while  $a$  is in its interior. Now we colour the vertices of  $D$  in the interior of the cycle by using colour 1, and those in the exterior by 2.

We claim that the above gives a 2-colouring of  $D$ . To see this, assume that the process gives rise to a monochromatic directed cycle  $R$ . Then we may assume that the whole cycle  $R$  lies in the interior of  $C^*$ . Since this is also a cycle in  $G$ , its interior contains a vertex of  $H$ , but since  $C^*$  is a Tutte path, any such cycle separates a  $C^*$ -component with at most 3 attachments. Since  $G$  is 4-connected, this corresponds to a vertex in  $H$  and the cycle  $R$  is a triangle. But  $D$  has digirth at least 4, so  $R$  is not a directed cycle. This contradiction proves the claim.

**4-connected – one colour case.** Suppose now that all three vertices of  $T$  have the same colour. In this case we proceed in the similar way except that we consider the graph  $H$  obtained from the dual of  $G$  by deleting the vertex  $t^*$ . The resulting graph is 2-connected and vertices  $a^*, b^*, c^*$  all lie on the outer facial cycle  $C^*$ . Let  $u$  be a neighbor of  $a^*$  on  $C^*$  and  $w$  a neighbor of  $c^*$  in  $C^*$ . Now we take a Tutte path with respect to  $C^*$  from  $a^*$  to  $u$  that passes through the edge  $e = c^*w$ . Together with the edge  $ua^*$  we obtain a cycle  $R$ , and we colour the vertices of  $D$  inside this cycle differently from the colour used on  $a, b, c$ ; and the vertices outside this cycle the same as  $a, b, c$ .

The above gives an extension of the colouring of  $T$ , as desired. We only need to argue that there are no monochromatic directed cycles of  $D$ . As before, the cycles of  $G$  that are monochromatic are triangles that are dual to vertices in 3-attached  $R$ -components. All of these correspond to facial triangles in  $G$  that are not directed cycles in  $D$  by the assumption on the digirth. The only possible difference from the first case might be an  $R$ -component containing the vertex  $b^*$  (when  $b^* \notin V(R)$ ). By property (ii) of Tutte cycles, this  $R$ -component in  $H$  is 2-attached. But in the dual graph of  $G$ , it is adjacent to  $t^*$  and thus gives a monochromatic subgraph of  $D$  that contains  $a, b, c$  and the vertex  $b' \neq b$  forming the second facial triangle with the edge  $ac$  of  $G$ . However, since all edges incident with  $a^*$  and  $c^*$  are in  $R$ , this subgraph only consists of four vertices. The two triangles in this subgraph are not directed cycles in  $D$ , which implies that also the 4-cycle in this subgraph cannot be a directed cycle. This shows that  $D$  has no monochromatic directed cycles.

**Non-4-connected case.** If  $G$  is not 4-connected, then there is a triangle  $T' = xyz$  that separates the graph, one component being the subgraph  $D_0$  in the exterior of  $T'$  (including  $T'$ , which now becomes a facial triangle) and the other one,  $D_1$ , formed from  $T'$  and the vertices and edges in the interior of  $T'$ . Now we first extend the precolouring of  $T$  to  $D_0$  (by using the induction hypothesis), and then apply the induction hypothesis to  $D_1$  to extend the colouring of  $T'$  obtained in the first step. By Lemma 2.2, the combined colouring is a 2-colouring of  $D$ .  $\square$

*Proof of Theorem 1.2.* We may assume the underlying undirected graph  $G$  is triangulated as otherwise, we can add a vertex inside each face of  $D$  adjacent to all vertices of that face in  $G$  and direct all edges towards the new vertices in  $D$ . This creates no new directed cycles.

The proof is now essentially the same as the proof of Lemma 2.3 with one difference the we will explain below. If  $G$  has a separating triangle, the induction step is the same as in Lemma 2.3 (by applying either the lemma or this theorem inductively). So, it suffices to consider the 4-connected case, and we are not bound with any precolouring.

The faces of  $G$  containing  $v_0$  form a cycle  $C^*$  in the dual graph  $G^*$ . Let us choose three consecutive vertices  $u^*, v^*, w^*$  on  $C^*$ . A Tutte path with respect to  $C^*$  joining  $u^*$  and  $v^*$  and containing the edge  $v^*w^*$  forms a cycle  $R$  together with the edge  $u^*v^*$ . By the connectivity of  $G^*$ , each  $R$ -component with two attachments is an edge. So all edges of  $C^*$  are either an edge of  $C^*$  or an entire  $P$ -component by property (ii) of Tutte paths. In particular, no vertex of  $C^*$  is in a 3-attached  $R$ -component and hence their duals, triangles containing  $v_0$ , all have a vertex inside  $R$  and a vertex outside  $R$ . Thus, using colour 1 for all vertices inside  $R$  and using colour 2 outside  $R$  gives an acyclic colouring of  $D$ .  $\square$

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